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by

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Submitted in partial fulfillment of
the requirements for the degree of

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ABSTRACT

One of the foremost theoretical problems in operations research is that of choosing an appropriate measure of effectiveness. Since such a choice involves value judgments, the point of view is taken that the purpose and preferences of the subject (decision maker) should be used as the basis for defining the measure of effectiveness. Use is made of established results in the field of economics, specifically utility theory, to show that the ordinary notions of subjective preference are sufficient, under certain plausible conditions for a single subject, to define a measure of effectiveness which is unique up to a linear transformation. An illustrative example of naval interest is given.

This thesis was written during the period January - June, 1957, at the U. S. Naval Postgraduate School, Monterey, California. The writer wishes to express his appreciation to Professor C. C. Torrance for his encouragement, guidance, and many helpful suggestions during the writing of this thesis, to Professor C. A. Magwire for his constructive criticism while acting as second reader, and to Miss Bessie Allan for her invaluable clerical assistance in the preparation of the manuscript.

FOREWORD

The author feels compelled to offer a prelinimary explanation of the circumstances which led to the writing of this paper. The principle reasons are two-fold. First, inspiration was drawn from articles appearing recently in Operations Research, the Journal of the Operations Research Society of America, notably those authored by B. O. Koopman of Columbia University [1] and Charles Hitch of The Rand Corporation [2], pointing up certain logical failings so frequently encountered in the choice of measures of effectiveness, or criteria. Secondly, there has been a personal dissatisfaction with the techniques of problem formulation, particularly in the choice of measures of effectiveness. evidenced in many reports of operations research studies used as text and reference material in various courses of study at the United States Naval Postgraduate School.

The particular line of investigation of this problem was suggested by Hitch in writing

> The only discipline I know which has made any attempt to explore the characteristics of operations criteria ..... is economic theory.

By following this lead, a remarkable similarity was discovered between the problems of economists in attempting to develop a satisfactory theory of utility and the difficulties of operations researchers in dealing with the problem of choice of criteria.

Historically, economists first conceived of utility as quantitatively measurable, i.e, - a number. Now every claim of measurability must ultimately be based on some immediate sensation,[1]

[1] Such as light, heat. muscular effort, length, etc.

which need not and possibly cannot be analyzed further. In the case of utility, the immediate sensation of preference for one object or event as against another provided this basis. In economics Pareto first observed that an equality relation of utility differences would suffice to prescribe a measure of utility which would be unique up to a linear transformation. Exactly the same argument was made by Euclid in his classical derivation of numerical distances. The utility difference relation was objected to by many economists on the grounds that such a relation did not stem from a "natural" sensation. Von Neumann and Morgenstern offered a way around this difficulty by postulating that individuals could compare not only objects or events but also combinations of objects or events when expected with certain stated probabilities. Even this suggestion has not found universal acceptance and the question of measurability of utility remains unresolved. The modern method of indifference curve analysis is a mathematical method to describe the situation in which the concept of utility differences is not admitted.

Like the economists' classical concept of utility, operations researchers traditionally regard the measure of effectiveness as numerical. But upon what sensation is it based? This question, it is believed, has not been adequately answered and it is suggested that herein may lie solutions to some of OR's problems.

In this thesis the viewpoint is taken that, as with economic utility, the measure of effectiveness should be based upon the "natural" sensation of preference, specifically the preferences of the decision maker, executive, client, etc. referred

to herein as the "subject". After an introductory section this viewpoint is, it is hoped, somewhat justified in Sections 2 and 3. In order to retain the numerical character of the measure of effectiveness, the notions of utility as set forth by von Neumann and Morgenstern and extended by Dalkey, are adopted. In Sections 4 through 6 these notions are interpreted in a form more suitable for OR and a measure of effectiveness (value function) derived which is mathematically equivalent to the von Neumann-Morgenstern individual utility function. The value function derived is however more general in that it allows for moral, aesthetic, military, or any other sense of value as well as the strictly utilitarian sense. In Section 7 an example of the application of these concepts to a hypothetical problem of naval interest is presented.

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LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

(Listed in order of use in the text)

x	an alternative (also any lower case English letter, e.g. - y, u, v, etc.)
R	a binary relation between alternatives (preference or indifference)
P	a binary relation between alternatives (preference)
I	a binary relation between alternatives (indifference)
$[x,y,\alpha]$	an option
$\sim$	a binary relation between alternatives (comparability)
$\mid$	a binary relation between alternatives (incomparability)
V	a value function
C	an indifference class of alternatives
$\overline{O}$	the objective
$\underline{O}$	the origin (opposite of objective)

1. Introduction

A typical operations research study may be described as consisting of three phases -- formulation of the problem, solution of the problem, and implementation of the solution. Such a description is functional rather than sequential, and by no means unique. For example, factors arising in the solution or implementation phases may well lead to a reformulation of the problem. However the description is convenient in defining the scope of this thesis -- the formulation of an operations research problem and, specifically, the choosing of a measure of effectiveness. It is not concerned with techniques of solution nor with aspects of implementation.

An operations research study begins naturally with problem formulation, at least an initial formulation. It is much like defining the rules of a game before play commences. The rules of any game will include among other things:

(a) an objective

(b) a method of scoring

(c) a statement of allowed plays or actions

By "an objective" is not meant the customary one of winning, but rather that which constitutes winning, e.g. - having the highest (or lowest) score at the end of a certain period of time, accomplishing a certain feat before one's opponent, etc. The method of scoring may range from a simple "win or lose" to a complicated scheme of point assignment for certain achievements. Actions or plays may be limited by physical boundaries and by specifying the types of action permitted and prohibited within the physical bounds. It should be noted that game rules are

not rigid but are subject to change by mutual agreement of participants. In organized sports rules committees meet periodically to consider proposed changes to rules, but until such a change is made all participants are governed by existing rules.

There is a close analogy between the defining of the rules of a game and the formulation of an operations research problem. In formulating an OR problem there should be specified among other things:

(a) an objective or purpose

(b) a measure of effectiveness

(c) alternative courses of action

In specifying the objective in an OR problem, as in a game, a more precise statement than simply "best results" or "optimal outcome" (i.e., - winning) should be made. The objective is that which constitutes the best or optimum under the given circumstances. The measure of effectiveness is analogous to the method of keeping score in a game. It should assign to a given outcome (result of a play) a unit of worth or value. The alternative courses of action correspond to the allowed plays in a game. Just as in a game certain plays are favored under certain circumstances because they tend to bring about the winning state, in an operations research study certain courses of action will be determined to be better because they are expected to bring about the achievement of the objective or that (feasible) outcome which has associated the highest worth or value.

Certain aspects of the above analogy deserve further emphasis. Consider first the relation between the objective (purpose) and the method of scoring (measure of effectiveness). The score

measures directly the achievement of objective. This holds true even if the concept of objective is generalized beyond the narrow one of winning. The purpose in a given contest may not be to win but to "hold the score down", such as when a football team plays a vastly superior opponent. The score, even in this case, provides a direct measure of achievement of objective. Such should be true with a measure of effectiveness. Perhaps the most desirable outcome is unattainable because of technological or other constraints. In such a case the measure of effectiveness should still provide a direct measure of achievement of objective, at least in the sense of how nearly it is approached.

Next consider the relation between method of scoring and allowed plays or actions. Score in a game is (usually) not related directly to plays or actions but rather to the outcomes which result. For example, there is no award of points in football for attempting a forward pass or even for the successful completion of one. Points are awarded only if <u>as a result</u> the necessary yardage is gained to score a touchdown. However, the allowed actions are related, at least stochastically, to the outcomes so that, for a given play attempted, it should be possible to determine the expectation of score. Similarly a measure of effectiveness is not necessarily related directly to an alternative course of action but rather to the outcome or state resulting from such a course of action. It is by estimating the expected outcome of a given course of action that a value can be assigned to the various courses of action. It should be noted here that it is the role of the mathematical model (or other technique of problem solution) to estimate the outcome

which will result from selecting a given course of action. Indeed it is the various controllable variables (parameters) involved in the mathematical model which determines the feasible courses of action.

A problem of theoretical concern in operations research, and increasingly so, is that of selecting an appropriate measure of effectiveness. The difficulties encountered in attempting to associate a real number with factors that have no "natural" scale of measurement (e.g. - good will, satisfaction, etc.) are well known. Equally difficult are the problems in which the objective is multidimensional and the various factors are "incomparable" (e.g. - loss of human life versus attainment of a military objective). As discussed in the introduction, the problem of selecting a measure of effectiveness has a parallel in the concept of "utility" in the field of economics. Like a measure of effectiveness utility was at first conceived as quantitatively measurable, i.e., as a real number. Clearly every claim of measurability must be based on some immediate sensation, which possibly cannot (and need not) be analyzed further. In the case of utility the immediate sensation of preference provides this basis. Preference, however, only permits it to be said that for one person the utility of one object or situation is greater than the utility of another such object or situation. It does not by itself provide an adequate basis for measurability. It is at this point that economists diverge widely in viewpoints. Some contend that there is no other "natural" sensation appropriate to the problem than that of preference, hence "measurable utility" is not admitted. The method of indifference curve

analysis is a mathematical procedure to describe this situation; Von Neumann and Morgenstern hypothesized the existence of a natural operation rather than another natural sensation -- the ability to distinguish (in the sense of preference) between an alternative and two others expected with certain probabilities. With this hypothesis and assuming the complete ordering by preference of any set of alternatives it was shown that a numerical utility was determined which was unique up to a linear transformation. Dalkey, extending the work of von Neumann and Morgenstern, showed that the introduction of this type of expectation (which he termed "options") made unnecessary the assumption of complete ordering by preference, and thus made the results applicable to a wide class of partial orderings, i.e. - multi-dimensional vector quantities.

In this thesis the results of von Neumann and Morgenstern and Dalkey are translated to the problem of determining a measure of effectiveness in an operations research problem, particularly those in which the objective is multi-dimensional.

2. Optimality and Subjectivity

One of the most commonly used words in the field of operations research is the word "optimal". A typical dictionary defines [3]

Optimal - best, most favorable or most conducive to a given end, esp. under fixed conditions.

It is to be noted that, as defined, optimality is not intrinsic but depends upon the end or purpose involved. Before the optimal can be determined the end or purpose must be specified. The meaning of these words is embodied in the definition [3]

Purpose - that which one sets before himself as an object to be attained, the end or aim to be kept in view in any plan, measure, exertion, or operation.

This definition, by no means unique, implies that purpose is subjective rather than objective by nature. However the nature of purpose is a question which properly belongs to philosophy rather than science.

Philosophy, which from the Greek means "love of wisdom", differs from science in that both the natural and social sciences base their theories on established fact, whereas philosophy covers the area of inquiry where, essentially, no facts are available. Originally science as such did not exist and philosophy covered the entire field. But as facts become available and tentative certainties emerged, the sciences broke away from metaphysical speculation to pursue their different aims. Thus physics was once a realm of philosophy and it is only recently that such fields as psychology have been established as sciences apart from philosophy. The concern of modern philosophers (Dewey,

et al) in attempting to develop a theory of value shows promise that questions of purpose and related matters may someday develop into a true science, but for the present such is not the case. [4]

In an operations research study an initial concern is the specification of an objective, i.e. - the determination of purpose. However since the nature of purpose is a philosophical conjecture, the scientific treatment of an operations research problem requires that the nature of purpose be hypothesized. In this thesis it is hypothesized that purpose is subjective by nature. This point of view is adopted for two reasons. First, a number of theoretical difficulties which arise from an objective viewpoint of purpose are avoided. Secondly, the idea is intuitively appealing that the desires of the person(s) variously termed the client, decision maker, executive, customer, etc. are the proper basis for determination of purpose rather than the ideas of an analyst or some other person not charged with the responsibility for results of a decision.

An hypothesis of subjective preference does not conflict with the traditional demands of science that facts be publicly observable and verifiable. Nor does it imply that the analyst should let the customer define the problem. It means that in a given problem the analyst should specify the objective only after suitable observation and testing, if necessary, of the subject of the problem (i.e. - the client, decision maker, etc.).

Although an assumption of the subjective nature of purpose has certain justifications, even a superficial investigation from such a starting point soon encounters difficulties. Perhaps

the most disturbing is the problem of determining the objective of a group. This question has been treated extensively by Arrow and others in developing the theory of welfare economics. [5] A social welfare function is defined as a method for obtaining group preferences given the preferences of the invidual members of the group. An election system, for example, gives a "community choice" of candidates as a function of voters' choices. Arrow has shown (1) that if individual preferences are expressed as rankings of various alternatives and (2) that if we require certain "natural" conditions of any "acceptable" social welfare function, then "acceptable" welfare functions do not exist. It is shown that, under certain plausible conditions, any welfare function is either "dictatorial" (group choice determined by one individual) or "imposed" (no real choice exists by virtue of some kind of constraint). Some writers have questioned the plausibility of Arrow's assumptions and, under different assumptions, have obtained more intuitively acceptable results. [6] A detailed treatment of this problem is beyond the scope of this thesis. Further it does not affect the large class of problems, common to operations research, wherein decisions are in fact made by an individual, i.e. - an executive, at least with advice of others of a group. Therefore it will be assumed for the purposes of this paper that the subject of a problem is a single individual, or is a special group whose preferences can be determined as if it were an individual.

3. Preference Relations

In the preceding sections the formulation of an operations research problem has been discussed in a general way and the hypotheses of subjective purpose and singular subject have been adopted. To avoid ambiguity in the remaining sections the rather special meanings to be attached to certain terms will be formalized by explicit definition.

<u>D.1</u> <u>Alternative</u> - a state of a given system which is describable by an n-dimensional vector.

As defined an alternative is not a possible course of action but rather an outcome which might result from some course of action. The outcome must be describable by an n-tuple of real numbers. This is not such a restrictive requirement as it might at first appear. It does not, for example, require that an intangible factor such as "good will" be measured in dollar or other specific units; it requires only that some scale be established which distinguishes in a suitable manner among the different levels of "good will". Under these circumstances many such intangibles can be "measured" in some suitable unit. (The conditions under which this can be done are discussed in Section 5.)

A subject will frequently be able to choose between two alternatives, i.e. - to state or display a preference for one over the other. Sometimes indifference will be exhibited, i.e. - the two alternatives will be considered equally acceptable and neither is preferred to the other. Occasionally a subject will be unable to make a choice; he will be unwilling to choose one alternative in preference to the other and yet is not indifferent. It should not be expected that a subject always be able to judge

between two specified alternatives but a certain "consistency" or "rationality" is required intuitively. For example, a rational subject would not prefer A to B, B to C, and at the same time prefer C to A. A subject might however be indifferent among three or more alternatives.

Alternatives are considered as future conditions or events. There is no apparent reason why alternatives located in different time periods of the future should not be compared (i.e. - time is considered as one component of the n-dimensional vector which describes an alternative), however, such a concept would introduce a complication unnecessary for the purposes of this thesis. Therefore it will be assumed that, in considering any particular set of alternatives, all such alternatives are located at one and the same time, preferably in the immediate future.

Preference and indifference are relations among alternatives (see Appendix I). Instead of working with two relations it will be convenient to define a single fundamental relation, denoted R, and define preference and indifference formally in terms of R.

<u>D.2</u> R is a binary relation between alternatives. xRy means that a given subject confronted with a choice between two alternatives, x and y, will choose x <u>or</u> will consider x and y equally acceptable. xR'y means "not xRy".

<u>D.3</u> P is a binary relation between alternatives. xPy means:

(a) xRy

(b) yR'x

<u>D.4</u> I is a binary relation between alternatives. xIy means:

(a) xRy

(b) yRx

The relation R as defined is similar to the relation "greater than or equal to" in the space of real numbers; xRy is read "x is preferred or indifferent to y". The relation P is similar to the relation "greater than" in the space of real numbers; xPy is read "x is preferred to y". The relation I is similar to the relation "equals" in the space of real numbers; xIy is read "x is indifferent to y". It should be noted that I does not imply logical identity. Logical identity is denoted "$\equiv$".

It is postulated that the fundamental relation R is a partial ordering relation on any space of alternatives, S. Formally,

<u>P.1</u> (a) xRx for all $x \in S$

(b) xRy and yRz imply xRz

It can be readily shown from the postulated characteristics of R that P and I as defined correspond to the intuitive notions of preference and indifference. (See Appendix II).

<u>P.2</u> If $x,y \in S$ where $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$:

(a) xRy if $x_i R y_i$ for all $i=1,\ldots.,n$.

(b) xPy if xRy and $x_i P y_i$ for some i.

The implications of P.2 are illustrated by the case in which the n components of the alternative vectors have real numbers prescribed in such a way that, for each component, it is desirable to attain the largest possible number. The relation R, in such a case, corresponds exactly to the relation "greater than or equal to" and the relation P to the relation "greater than". Also from P.2 the conditions for incomparability are immediate; if xRy does not hold and yRx likewise does not hold, x and y are clearly incomparable with respect to R (see Appendix I).

An an example consider a military commander estimating his battle losses. Suppose, for simplicity, his losses can be measured in terms of equipment destroyed (say aircraft) and personnel lost (say the pilots). The state of the system of interest can be described by a two dimensional vector (i,j), where i is the number of aircraft lost and j is the number of pilots lost. If i and j are represented as negative numbers, we may assume that any (rational) commander would wish to maximize both i and j. The most desirable possible outcome (with respect to losses) would be vector (0,0). It may be assumed that any commander would prefer the outcome (-2,-2) to the outcome (-5,-5). However, these two outcomes are R-comparable. If such a comparison is attempted between other possible outcomes, say (-2,-5) and (-5,-2), a difficulty arises. Before preference can be determined some decision of relative worth of the two components under the given conditions must be made, i.e. - in some manner the two outcomes must be made comparable.

Having defined formally an alternative and a single fundamental relation, R, between alternatives, it is now possible to define concisely the concept of an objective.

<u>D.5</u> In a space of alternatives, S, if for a given subject:

(a) $x \in S$

(b) xPy for all $y \in S$, $y \not\equiv x$

x is the unique objective.

An objective is the most desirable possible outcome that could result from any course of action. It seems unnecessary to show formally the conditions under which an objective exists. It is

intuitively clear that, if each component of the alternative vectors is bounded by a finite maximum[1], the objective will exist and will be that outcome for which each component assumes it's maximum value.

[1] This assumes that a higher valued real number is to be preferred to a lower valued one. This can always be made the case by taking negative numbers to describe a component in which a lower valued (positive) real number would be preferred to a higher valued number. Mathematically, such a space of alternatives would be represented by a rectangular, closed, convex sub-space of Euclidean n-space.

4. The Complete Ordering of Alternatives

A single fundamental relation, R, has been defined and postulated to prescribe a partial ordering of any given space of alternatives considered by a given subject. In some simple cases, where the alternatives are one dimensional vectors, R will be a complete ordering relation, e.g. - the alternatives are describable as amounts of money. More commonly the alternatives will be multi-dimensional and their direct conversion to a single dimension will be difficult if not impossible. As exemplified in the preceding section, a military commander may be able to express the value of a lost aircraft, but his problem in estimating the worth of the lost pilot in the same units is obvious.

Under certain conditions the partial ordering relation R can be extended to a complete ordering relation in the case of multi-dimensional alternatives if it is assumed that a subject can judge between an alternative and a combination of two others, say u and v, where v is expected with probability α and u with probability $1 - \alpha$. This form of expectation will be called an option.

<u>D.6</u> An option is an exclusive disjunction of two alternatives where one must occur, denoted $[u, v, \alpha]$, to be read "v with probability α and u with probability $1 - \alpha$".

Just as it was assumed that a subject could not always judge between two alternatives, it will be assumed that a subject may not always be able to judge between an alternative and an option. The nature of options is given by:

P.3 (a) For all $x, y \in S$, $x \sim y$[1] implies $[x, y, \alpha] \in S$ where $0 \leq \alpha \leq 1$.

(b) $xRyRz$ implies the existence of an α, $0 \leq \alpha \leq 1$, such that $yI[x, z, \alpha]$.

(c) xRy implies $xR[x, y, \alpha]Ry$ where $0 \leq \alpha \leq 1$.

(d) $[x, [x, y, \alpha], \beta] \; I \; [x, y, \alpha\beta]$ where $0 \leq \alpha, \beta \leq 1$.

In P.3, (a) allows the combination of two comparable alternatives to form an option. In (b) the mutual comparability of three alternatives implies the existence of an option composed of the two "extremal" alternatives which is indifferent to the "middle" alternative. (c) states that an alternative is to be preferred (or indifferent) to any option involving the given alternative and a less desirable one. (d) determines how options combine which is, essentially, like mathematical expectation. In the remainder of this section it is shown that the introduction of options is sufficient under certain conditions to extend R to a complete ordering relation. In Section 5 it is shown that the introduction of options is sufficient to define a unique (up to a linear transformation) numerical measure, the value function over the indifference classes of a space of alternatives.

Before considering the sufficiency of options to extend R to a complete ordering it will be convenient to redefine several well known mathematical notions in forms suitable for this investigation.

D.7 x is a lower bound of a set, Q, of alternatives with respect to R means that for every $y \in Q$, yRx.

[1] The symbol "$\sim$" denotes comparability. See Appendix I.

D.8 x is an upper bound of a set, Q, of alternatives with respect to R means that for every $y \in Q$, xRy.

D.9 Q is a quasi-lattice with respect to R means:

(a) Q is partially ordered by R

(b) Every pair $x, y \in Q$ has an upper bound and a lower bound.

The following important theorem gives a condition under which a space of alternatives is completely ordered by the relation R. [7]

T.1 For a given subject, a space of alternatives S that is a quasi-lattice with respect to R and that satisfies P.3 is completely ordered by R.

Proof: R is reflexive, asymmetric, and transitive by D.9(a), hence only connexity need be shown to establish complete ordering of S by R. If $x \sim y$ for all $x, y \in S$ (e.g. - alternatives are one dimensional) the proof is complete, hence assume $x | y$ for some $x, y \in S$. Since S is a quasi-lattice with respect to R there is some u which is a lower bound of x,y and some v which is an upper bound of x,y. Then by D.7 and D.8 vRxRu and vRyRu. By P.1 vRu, hence $v \sim u$. By P.3 $[v,u,\alpha] \in S$, and there exist α, β such that $xI[v,u,\alpha]$ and $yI[v,u,\beta]$, hence $x \sim y$.

The implications of the above definitions and theorem may be illustrated by referring again to the example of the military commander estimating his battle losses (Section 3). The space of alternatives has as a lower bound the outcome $(-n_i, -n_j)$, where n_i is the total number of aircraft and n_j is the total number of pilots. Clearly there could be no worse outcome than total loss of both components. Similarly the upper bound of the space of alternatives is the outcome (0,0), i.e. - the objective, or no losses.

Also the space of alternatives is a quasi-lattice since every pair x,y has an upper bound and a lower bound, e.g. - the incomparable outcomes (-2,-5) and (-5,-2) have as (one pair of) upper and lower bounds respectively the outcomes (-2,-2) and (-5,-5). Since the space of alternatives is a quasi-lattice it is, upon the introduction of options, completely ordered by R.

5. Value

The nature of value, like the nature of purpose, is a question of philosophical concern. Whether value is objective (objects, etc. have intrinsic value apart from any personal relation), or subjective (objects, etc. have value only in respect to some personal relation), or some intermediary of these viewpoints is a matter of increasing interest in modern philosophy. As with purpose it will be hypothesized that value is subjective by nature. For the purposes of this thesis value will be defined as follows:

<u>D.10</u> <u>Value</u> - For a given subject and a given bounded space of alternatives, S, if there is associated with each $x \in S$ a real number $V(x)$ with the following properties:

(a) For all $x,y \in S$, xRy implies $V(x) \geq V(y)$

(b) For all $x,y \in S$, $V(x) \geq V(y)$ implies xRy

then $V(x)$ is called the value of x.

Such a definition is in harmony with a commonly accepted modern philosophy of science (logical positivism) which is based on an operational viewpoint. Measured quantities such as charge, temperature, mass, and force are not thought of as things whose nature is intuitively understood. They are defined as the objective results of certain prescribed operations that can be carried out in a laboratory by any experimenter. So it is with value as defined. Further, such a concept of value is more general than the usual concept of utility. No attempt is made to specify whether a particular choice is prompted by moral, utilitarian, aesthetic or other considerations; it is simply observed that the

choice is made. Theoretically it is possible to determine which of two alternatives has the greater value (to the subject) by confronting the subject with the two and observing which is chosen in preference to the other. There are, of course, many practical difficulties associated with such a proposed procedure. The determination of individual attitudes and preferences is a problem of concern in modern psychology. Unfortunately even a superficial treatment of such a field is beyond the scope of this thesis. It will therefore be assumed that, since such a procedure is at least theoretically possible, it can be carried out.

From the definition of value and P.3 it follows that if such a relation as xRy holds, then the relation $V(x) \geq V([x,y,\alpha]) \geq V(y)$, $0 \leq \alpha \leq 1$, must hold. Further since P.3 (a) admits any option formed by the combination of two comparable alternatives, i.e. - a "continuum" of options between the two alternatives, then there is a corresponding continuum of values between the values associated with two comparable alternatives. In other words the continuity of the space of alternatives (and options!) implies a continuity of the value function between the upper bound (objective) and the lower bound of the space.

Suppose that for a given subject and a given space of alternatives the following relations hold:

$$xRyRz$$

$$yR\,[x,z,0.5]$$

In other words, the subject prefers (or is indifferent to) the alternative y to a 50%-50% chance of obtaining x or z. Such a preference is a plausible basis for asserting that

$$V(x) - V(y) \leq V(y) - V(z)$$

If the sense of the R relation had been reversed then the sense of the inequality would also have been reversed, or if the subject had been indifferent, the value differences would have been exactly equal. The formation of options thus provides a plausible basis for the determination of value differences, or at least the relative magnitudes of such. This assertion is identical with that made by von Neumann and Morgenstern[1], and is sufficient for the determination of an essentially unique value function.

A completely ordered space of alternatives may, upon the introduction of options, be considered as a continuous one-dimensional set of indifference classes. It is desired to coordinate numbers, V, to the various indifference classes, C, of this set. This can, of course, be done in many ways. Let $V=V(C)$ be one such coordination, where $V(C)=V(x)$ for all $x \in C$ and such that if

$$C_1 R C_2 R C_3 \text{ and } C_2 R \left[C_1,\ C_3,\ 0.5\right]$$

then

$$V(C_1) \geq V(C_2) \geq V(C_3) \text{ and}$$

$$V(C_1) - V(C_2) \leq V(C_2) - V(C_3) \text{ for all } C_i.$$

Now suppose $V' = V'(C)$ is another such coordination and satisfies the same conditions as V. Then V' may be regarded as a mathematical function of V. This follows from the fact that for every indifference class C there corresponds one and only one number V and also one and only one number V'. Consequently to every number V there corresponds one and only one number V'.

[1]For a more detailed treatment of the justification and implications of such an assertion the reader is referred to "The Notions of Utility", Ch.I, Section 3 [8]

Let this correspondence be denoted $V' = f(V)$. Furthermore from $V_1 \geq V_2 \geq V_3$ it follows that $V_1' \geq V_2' \geq V_3'$, and from $V_1 - V_2 \leq V_2 - V_3$ it follows that $V_1' - V_2' \leq V_2' - V_3'$. This means the function $V' = f(V)$ must satisfy the same conditions regardless of what numbers V_1, V_2, and V_3 may be. From this and the continuity of the function $f(V)$ it follows that if

$$V_1 - V_2 = V_2 - V_3$$

then

$$f(V_1) - f(V_2) = f(V_2) - f(V_3)$$

This may be restated as follows: if V_2 is the arithmetic mean of V_1 and V_3

$$V_2 = \tfrac{1}{2}(V_1+V_3)$$

then the value of the function $f(V)$ at V_2 is the arithmetic mean of the values $f(V_1)$ and $f(V_3)$, i.e. -

$$f(V_2) = \tfrac{1}{2}\left[f(V_1) + f(V_3) \right]$$

The only continuous functions $f(V)$ satisfying this condition for any two numbers V_1 and V_3 are the linear functions

$$V' = f(V) = aV + b$$

where a and b are arbitrary constants. The arbitrariness of the constant b means that the origin of the scale is undetermined; the arbitrariness of the constant a means that no absolute size is given but only relative size (unit of scale undetermined).

In summary, the sensation of preference allows recognition of greater value and options permit the determination of value differences. These conditions determine a value function which is unique except for origin and unit of scale.

6. The Measure of Effectiveness

Before discussing the measure of effectiveness it will be well to summarize briefly the results of preceding sections. A value function, V, has been determined which is unique except for origin and unit of scale provided that the "natural"sensation of (singular subject) preference is adopted as a basis and the operation of forming options is admitted. If these two principle hypotheses are accepted then V is essentially determined.

The role of the measure of effectiveness in an OR study has been described as that of measuring the achievement of an objective, or how nearly it is achieved. Clearly this is exactly what the value function V accomplishes for objectives as defined by D.5. Therefore, for a particular problem (given subject and space of possible outcomes or alternatives) it is plausible to adopt the value function V as a measure of effectiveness. It is apparent that if the subject is changed or a different space of alternatives is considered, then the function V will very likely be modified. It is the role of the mathematical model (or other technique of problem solution) to establish suitable bounds upon the considered space of alternatives and, in fact, to determine such questions as the dimensionality of the space. This illustrates again the essential interrealatedness of the various phases of an OR problem: the mathematical model employed for solution determines the feasible courses of action and predicts the expected outcomes, i.e. - defines a space of alternatives to be considered for various courses of action, and the measure of effectiveness, V, defined over the prescribed space of alter-

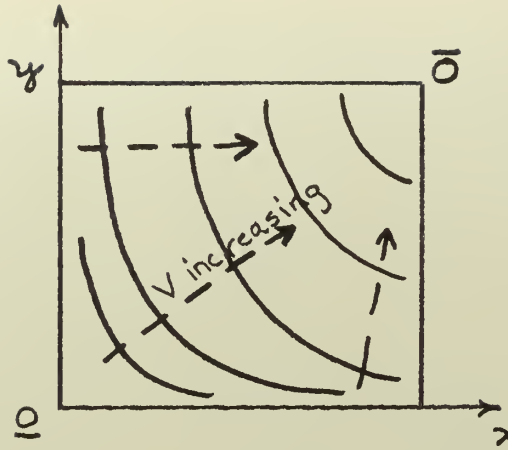
natives determines the objective and the relative desirability of the other possible outcomes or alternatives.

Certain characteristics of a measure of effectiveness are expected a priori, regardless of subject or the particular space of alternatives involved. Consider, for example, a given space of alternatives of dimensionality two. (Dimensionality two is considered for simplicity but the generalization to higher dimensionality will be evident). Suppose further that the two dimensions are not directly comparable, but that for each a higher real number descriptive of system state is to be preferred to a lower number. This can always be so arranged since, if a higher number is in fact less desirable than a lower number, such as some type of cost or loss, the negative can be taken and the situation reversed. Such a space may be plotted as a two dimensional vector space as in Figure 1. Clearly the objective will be the point indicated $\bar{O}$ and the least desirable state (the origin) will be the point indicated by $\underline{O}$. The concavity of the contour lines of equal value (indifference curves) toward the origin, $\underline{O}$, would be expected from the following considerations. If the bounds are chosen sufficiently large, for a small y and large x a saturation effect occurs as x is increased and an increase in V is achieved most readily by a small increase in y. In other words x becomes so plentiful that very little value is attached to obtaining more - the scarce component y is most desired. Similarly the greatest increase in V is obtained by moving in the x direction when x is small and y is large. The exact shape of the contour lines of equal value will, of course, vary widely under different circumstances but will remain in general con-

cave toward the origin if the bounds are sufficiently large.

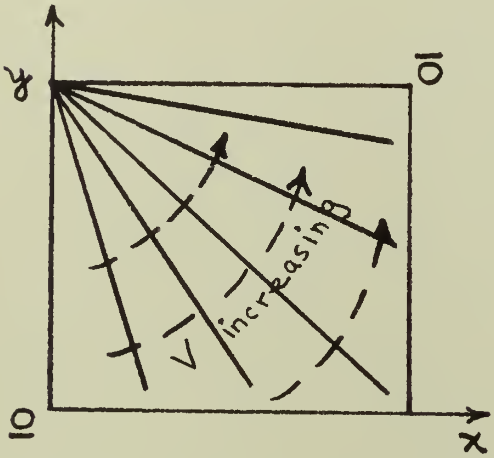
The importance of the above characteristic is perhaps best illustrated by comparing the (concave) contour lines of equal value for V with two commonly employed types of measures of effectiveness - the ratio or "exchange rate" type and the linear weight type. In the ratio type the ratio of unit effort per unit return is taken as the measure of effectiveness, e.g. - dollar cost per kill, etc. This measure is plotted in Figure 2 showing lines on which the ratio is constant. In the linear weight type the two "incomparable" components are assigned relative weights and a function of the form (ax+by) is taken as the measure of effectiveness. The choice of the weighting factors a and b may, of course, be a very difficult problem in itself. In Figure 3 this measure is plotted showing lines where the function (ax+by) is constant.

It is apparent by comparing Figures 1 through 3 that in certain regions the ratio or linear weight type may approximate V very closely (particularly if the weights are suitably chosen), but there will always be some region in which both types represent a direction of increasing value which is orthogonal to the direction in which the value function V increases. This is equivalent to saying that the absolute magnitudes involved are ignored by the ratio and linear weight types of measure of effectiveness. It is incumbent that such be taken into consideration when either of these types is used to approximate V.



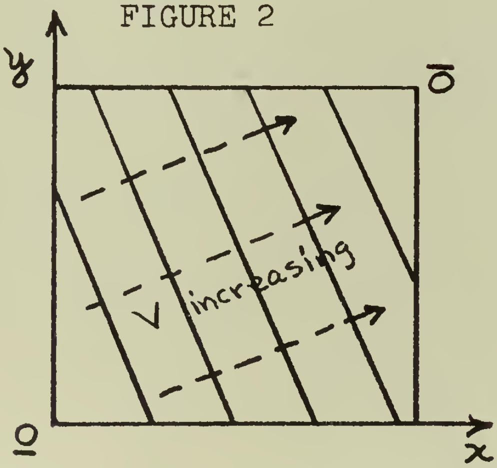
EQUAL VALUE CONTOURS - V BASED ON PREFERENCE

FIGURE 1



EQUAL VALUE CONTOURS - V BASED ON RATIO TYPE MEASURE OF EFFECTIVENESS

FIGURE 2



EQUAL VALUE CONTOURS - BASED ON LINEAR WEIGHT TYPE MEASURE OF EFFECTIVENESS

FIGURE 3

7. An Example

As an hypothetical example, but typical of problems encountered in military operations research studies, suppose that the U. S. Navy has the problem of developing and procuring an anti-submarine weapon system. The system is to be procured during time of peace for use against a potential enemy in the event of war. Suppose further that the possible outcomes of selecting any of several feasible systems can be described by a vector of two dimensions but that the two components are not directly comparable. Specifically, suppose that the two components are dollar cost which is subject to a peacetime budgetary constraint, and the expected return from a particular system which can be expressed, say, in number of enemy submarines destroyed during the period of interest. Now the space of alternatives is bounded by the budgetary limitations on expenditure and the number of submarines which the enemy is capable of producing during the time period of interest. Adoption of an appropriate mathematical model should permit the estimation of the cost of procurement and operation of the system and the expected return of the system expressed in number of submarines destroyed. These two quantities express the possible outcomes which might result from the selection of any particular feasible anti-submarine weapon system.

This hypothetical situation was presented to the second year students of the Operations Analysis curriculum at the U. S. Naval Postgraduate School, Monterey, California, and they were asked, individually, to express their preferences between various possible outcomes which might result under such circumstances. The detailed instructions given are included as Appendix III.

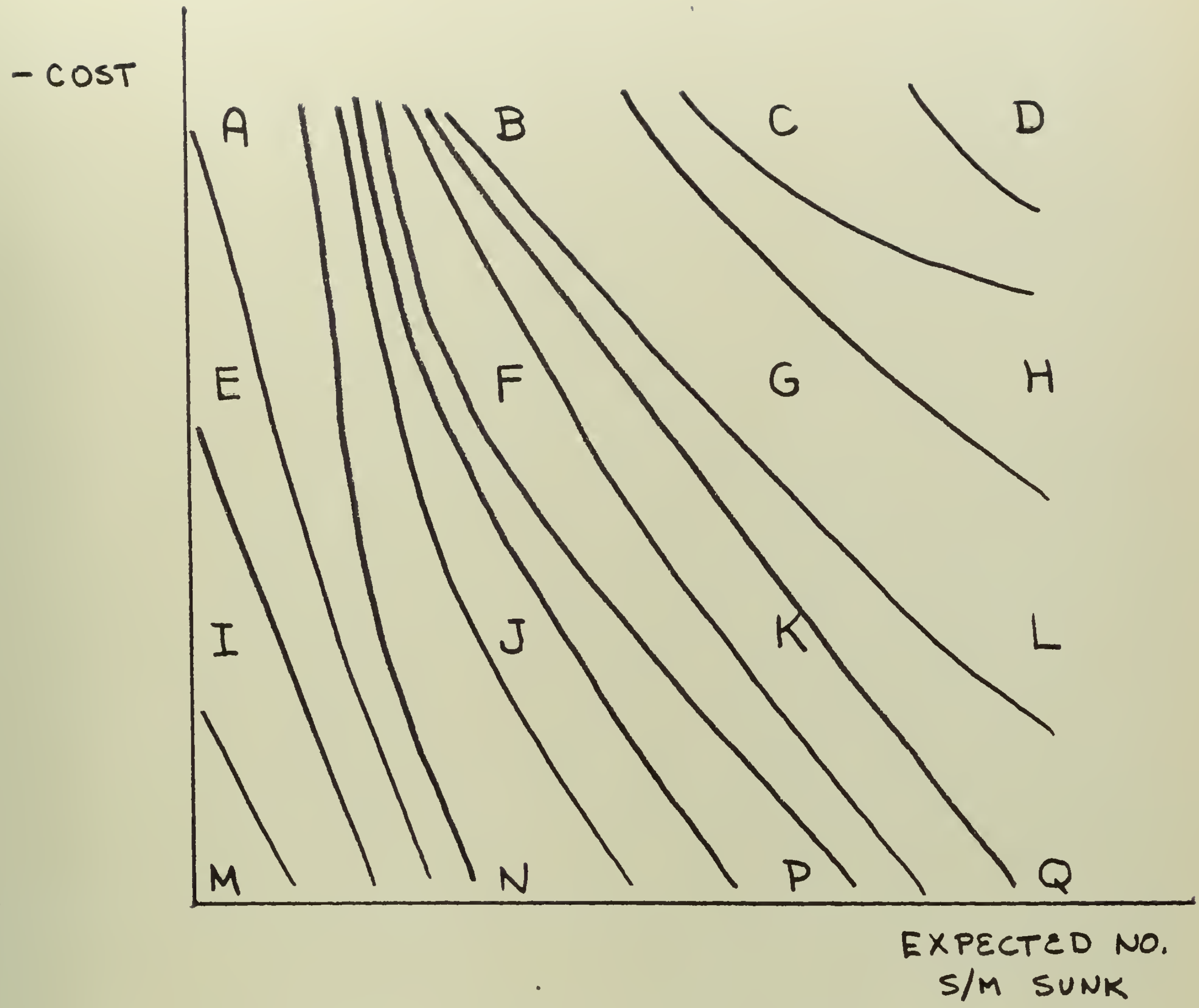
As an initial trial thirty eight choice pairs were given. These proved insufficient to approximate the value function V. The basic thirty eight were then supplemented as necessary for each subject in order to complete the approximate determination of V. As anticipated the function V differed with each subject though there were certain similarities among them all. The contour lines of equal value for a representative subject are plotted in Figure 4. Even in such an artificial situation the anticipated concavity feature was in evidence. In order to construct a specific value function the following conditions were assumed:

$$V(\bar{O}) = 1$$

$$V(\underline{O}) = 0$$

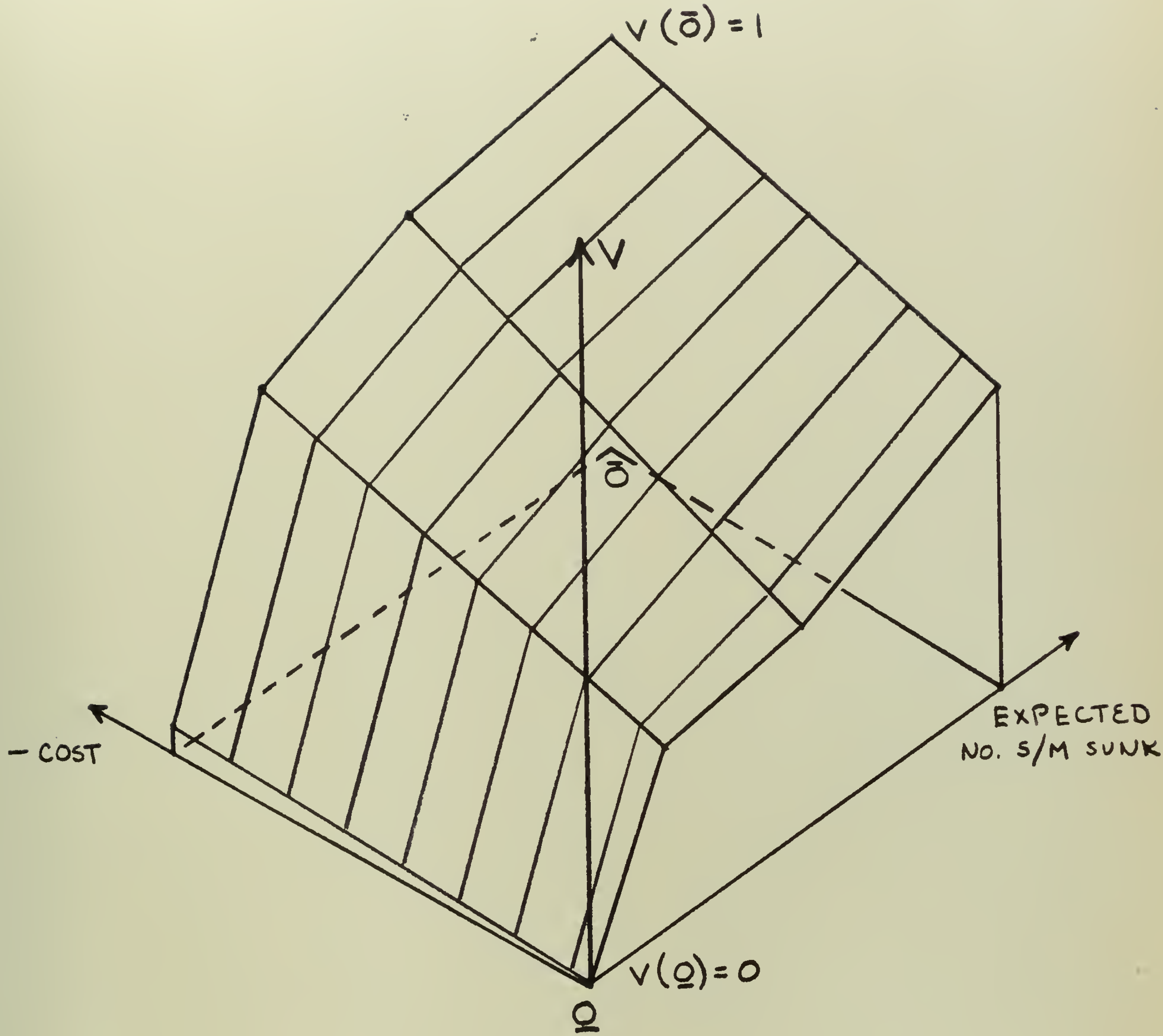
The resulting value function is shown in Figure 5.

Theoretically such a procedure can be employed in choosing a measure of effectiveness for a wide class of operations research problems. Practically, of course, there may be a number of difficulties. In an organization such as the United States Navy the mere identification of the subject may be a problem within itself. The techniques to be employed in determining the preferences of the subject, e.g. - construction of tests, would require much more careful consideration than given in the above hypothetical example. However such difficulties seem unavoidable if the sensation of preference is to be adopted as a basis for a numerical measure of effectiveness. Rejection of the numerical character of the measure of effectiveness would be most unsatisfactory. Yet if preference is not adopted as the basis for measure of effectiveness, the (perhaps) more formidable difficulty of specifying a different and adequate basis for measurement must be resolved.



PREFERENCE RELATION EXPERIMENT - EQUAL VALUE CONTOURS

FIGURE 4



PREFERENCE RELATION EXPERIMENT - VALUE FUNCTION

FIGURE 5

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A space S is said to be partially ordered by a relation O if O is a reflexive, asymmetric, and transitive binary relation between elements of S. If O is, in addition, connected, S is said to be completely ordered by O.

For a more detailed treatment see any standard text on advanced algebra. [9]

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APPENDIX II

CHARACTERISTICS OF PREFERENCE AND INDIFFERENCE

From the definition of the single fundamental relation R and its postulated characteristics (D.2 and P.1), intuitive notions of the relations P (preference) and I (indifference) as defined (D.3 and D.4) are readily proved, e.g. - the following characteristics:

(a) xPy and yPz imply xPz (transitivity of P)

(b) xPy and yRz, or xRy and yPz imply xPz (transitivity of P and R in sequence)

(c) xIy and yIz imply xIz (transitivity of I)

(d) $x \equiv y$ implies xIy (reflexivity of I)

Proof:

(a) xPy implies xRy by D.3. Suppose zRx; since xRy it follows that zRy by transitivity, P.1(b). But also by hypothesis yPz which implies yRz by D.3. Since the supposition leads to a contradiction it may be asserted that zR'x. Now by D.3 and P.1(b) xRz, hence xPz.

(b) Both hypotheses imply xRz by D.3 and P.1(b). Suppose zRx. zRx and yRz imply yRx by P.1(b), but this contradicts the (first) hypothesis that xPy which implies yR'x. Similarly zRx and xRy imply zRy, but this contradicts the (second) hypothesis that yPz which implies zR'y. Since both hypotheses are contradicted by the supposition that zRx, it may be asserted that zR'x which by D.3 and the previously proven xRz means xPz.

(c) By D.4 the hypothesis means xRy and yRx, and also yRz and zRy. By P.1(b) it follows that xRz and zRx which means xIz by D.4.

(d) Since $x \equiv y$ by hypothesis, it follows that xRy and yRx by P.1(a), hence xIy by D.4.

APPENDIX III

PREFERENCE RELATION EXPERIMENT

1. Situation As a naval planner you must select from several available an anti-submarine weapon system. The chosen system, selected in time of peace, is expected to be employed against a potential enemy in the event of hositilties. Assume the following conditions:

(a) Cost of a system includes procurement and operation during the useful life of a system.

(b) Useful life of all available systems essentially equal.

(c) Cost of a system can be expressed in dollar units and is subject to a budgetary constraint.

(d) Money not expended for this purpose will be available for other purposes.

(e) The enemy submarine potential during the useful life of the systems can be estimated with good accuracy.

(f) The probable damage which can be inflicted on the enemy submarine force by a given weapon system during its useful life can be estimated with good accuracy.

2. Instructions On your work sheet the possible outcomes of the above situation are depicted as two dimensional vector quantities. The two components are system cost in dollars and expected damage to the enemy in units of submarines destroyed. A series of choice pairs are listed. For each choice pair indicate which choice is considered preferable by circling the preferred choice. If the choices are considered equally acceptable circle both choices. Note that the second listed choice of each pair is an

option, i.e. - two possible outcomes are shown each of which will occur with probability 1/2. The first of each choice pair may be considered as certain to occur.

Assume budgetary limitations of one billion dollars and an enemy submarine potential of about one hundred submarines.

WORK SHEET

Cost (Billions of Dollars)

.25	A	B	C	D
.50	E	F	G	H
.75	I	J	K	L
1.0	M	N	P	Q
	0	25	50	75

Expected Number Submarines Sunk

In each of the following pairs, circle the preferred choice.

Pair	1st Choice	2nd Choice	Pair	1st Choice	2nd Choice
1	F	(G,J)	20	L	(H,K)
2	G	(D,J)	21	N	(J,M)
3	Q	(L,P)	22	K	(G,J)
4	J	(F,I)	23	F	(G,E)
5	H	(D,G)	24	F	(B,J)
6	B	(C,F)	25	G	(H,F)
7	F	(B,E)	26	G	(C,K)
8	J	(G,M)	27	J	(K,I)
9	G	(H,K)	28	J	(F,N)
10	F	(C,I)	29	K	(L,J)
11	I	(J,M)	30	K	(G,P)
12	K	(L,P)	31	B	(C,A)
13	P	(K,N)	32	N	(P,M)
14	K	(H,N)	33	C	(D,B)
15	A	(B,E)	34	P	(Q,N)
16	G	(C,F)	35	E	(A,I)
17	J	(K,N)	36	H	(D,L)
18	C	(D,G)	37	I	(E,M)
19	E	(F,I)	38	L	(H,Q)